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S-function series

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Abstract. We describe the different methods of generating S-function series, giving some new series as an example in each case.

1. Introduction

S-function techniques, though introduced in the last century by Cauchy and Jacobi, are still a lively branch of contemporary physics (King 1975, Yang and Wybourne 1986). As stated by Cummins and King (1987), they are 'an invaluable aid to calculations involving finite-dimensional representations of the classical Lie groups'.

More generally, the theory of symmetric functions is a rich field which has developed since the end of the eighteenth century starting with classical elimination theory (see Sylvester (1973), for example, for work from 1837-93). Nowadays, the convenient point of view, which we summarise in § 2, is to look at S functions as operators on the ring of polynomials and to use the λ-ring structure of this ring. Natural transformations then allow us to get, from one identity, an infinite number of seemingly different ones (see (2.3)-(2.7), (6.2) and (6.3)).

In § 3, we discuss some Littlewood-type formulae, enlarging the list of Yang and Wybourne (1986, § 4).

One basic and defining property of S functions is the Cauchy formula (4.1) which is very often not ascribed to this (French) author. From it, one recovers very easily the identities given by Yang and Wybourne (1986, § 5), to which we add (4.6) and (4.8) as further examples.

These manipulations do not add anything substantial to the work of Littlewood (1950). We present and illustrate in § 5 a more powerful method of symmetrising operators to generate S-function series (see (5.9) and (5.10)).

As a final comment, we give a determinantal expression of the plethysm which does not use the characters of the symmetric group (6.4).

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2. S functions

To any formal series $f=\Sigma_{k\geq 0} z^k S_k$, the S_k belonging to a commutative ring, Littlewood (1950, ch 6.4) associated the infinite matrix $\mathbb{S}(f)=(S_{j-i})_{j,i\geq 0}$, putting $S_i=0$ if $i<0$, and defined skew Schur functions to be the minors of this matrix.

More precisely, given $I=(i_1,\ldots,i_n)\in\mathbb{Z}^n$, $J\in\mathbb{Z}^n$, he defined $S_{J/I}$ to be the minor of $\mathbb{S}(f)$ taken on rows $i_n, i_{n-1}+1,\ldots,i_1+n-1$ and columns $j_n, j_{n-1}+1,\ldots,j_1+n-1$ (the minor is 0 if one of these numbers is less than 0). When $I=0^n$, one writes S_J instead of $S_{J/0^n}$. Of course, if the determinant $S_{J/I}$ has no zero column or row, nor identical columns (or rows), then by a proper reordering, one transforms $S_{J/I}$ into $S_{K/H}$, with H, K partitions (i.e. $k_1\geq k_2\geq\ldots\geq k_n\geq 0$). Indeed, one performs the reordering through the rule

$$S_{\ldots jj'\ldots/I}=-S_{\ldots j'-1\,j+1\ldots/I}. \tag{2.1}$$

Let $\mathbb{X}=\{x_1,x_2,\ldots\}$ be a finite or infinite set of variables. The original Schur functions, due to Cauchy, Jacobi, etc, are the ones associated with a polynomial $f=\Pi_i\,(1+zx_i)$, or with the inverse of a polynomial: $f=1/\Pi_i(1-zx_i)$. In the last case, one usually writes $S_{J/I}(x_1,x_2,\ldots)$ or $S_{J/I}(\mathbb{X})$.

One gets into problems of notation when one has to tackle simultaneously Schur functions associated with different series. The most efficient and compact way to proceed is to use the set-up of λ rings (Macdonald 1979, appendix), which amounts to considering the S_j to be operators on the ring of polynomial functions with real coefficients. This is done by associating with any polynomial $P=\alpha u+\beta v+\ldots$, with $\alpha,\beta,\ldots\in\mathbb{R}$, $u, v,\ldots$ monomials, the series $f_P(z)=(1-zu)^{-\alpha}(1-zv)^{-\beta}\ldots$. In other words, the $S_j(P)$ are defined through the generating function

$$\sum z^j S_j(P)=(1-zu)^{-\alpha}(1-zv)^{-\beta}\ldots. \tag{2.2}$$

Thus, when f is a generic rational series: $f(z)=\Pi_i\,(1-zy_i)/\Pi_j\,(1-zx_j)$ (one can write in short, following the notation of Einstein and Zweistein (1903), $f=\Pi_{x\in\mathbb{X},y\in\mathbb{Y}}(1-zy)/(1-zx)$), i.e. when $u, v,\ldots$, are variables and the coefficients $\alpha,\beta,\ldots$, are ± 1, the associated Schur function $S_{J/I}$ is now written $S_{J/I}((x_1+x_2+\ldots)-(y_1+y_2+\ldots))$ or $S_{J/I}(\mathbb{X}-\mathbb{Y})$. Accordingly, one has to identify a set of variables $\mathbb{X}=\{x_1,x_2,\ldots\}$ with the polynomial $\mathbb{X}=x_1+x_2+\ldots$.

The functions $S_{J/I}(\mathbb{X}-\mathbb{Y})$ are called by some people super-S functions, but in fact are a special case of Littlewood S functions attached to general formal series.

One can act separately on z and P. In particular, one has the multiplication of z by a scalar λ:

$$\boldsymbol{\lambda}: f_P(z)\to f_P(\lambda z) \tag{2.3}$$

and the multiplication of P by -1:

$$f_P(z)\to f_{-P}(z)=1/f_P(z). \tag{2.4}$$

Notice that $f_P(-z)\neq f_{-P}(z)$, and that (2.4) is an involution.

The associated Schur functions are transformed according to the following rules:

$$\boldsymbol{\lambda}: S_{J/I}(P)\to\lambda^{|J|-|I|}S_{J/I}(P) \tag{2.5}$$

where $|J|=j_1+j_2+\ldots+j_n$, and when J and I are partitions:

$$S_{J/I}(-P)=(-1)^{|J|-|I|}S_{J\sim/I\sim}(P) \tag{2.6}$$

where $J\sim$, $I\sim$ are the partitions conjugate to J, I (Macdonald 1979 I.1).

For example, the coefficients d_I appearing (see (4.8)) in the expansion $\Pi_{x\in\mathbb{X}}(1+x+x^2)=\Sigma\, d_I S_I(\mathbb{X})$ also give, thanks to (2.3) with $\lambda=-1$ and (2.4), the other three expansions

$$\prod_{x\in\mathbb{X}}(1-x+x^2)=\sum(-1)^{|I|}d_I S_I(\mathbb{X})$$
$$\prod_{x\in\mathbb{X}}(1-x+x^2)^{-1}=\sum d_I S_{I^\sim}(\mathbb{X})$$
$$\prod_{x\in\mathbb{X}}(1+x+x^2)^{-1}=\sum(-1)^{|I|}d_I S_{I^\sim}(\mathbb{X}).$$

The addition $P+Q$ corresponds to the product of corresponding matrices or series:

$$f_{P+Q}(z)=f_P(z)\cdot f_Q(z) \qquad \mathbb{S}(f_{P+Q}(z))=\mathbb{S}(f_P(z))\cdot\mathbb{S}(f_Q(z)).$$

Therefore, the Binet–Cauchy formula for the minors of a product of matrices (Muir 1812, ch IV) implies

$$\forall I,J\in\mathbb{Z}^n \qquad S_{J/I}(P+Q)=\sum_H S_{J/H}(P)\cdot S_{H/I}(Q) \tag{2.7}$$

sum on all partitions $H\in\mathbb{N}^n$.

The product of Schur functions is given by the Littlewood–Richardson rule (Macdonald (1979, I.9); see also the more efficient rule in Lascoux and Schützenberger (1985a)) which admits the following special case due to the Italian geometer Pieri:

$$\forall J\in\mathbb{N}^n, \forall P, \forall r\geqslant 0 \qquad S_{1^r}(P)\cdot S_J(P)=\sum S_H(P) \tag{2.8}$$

sum on all H: $(h_1,h_2,\ldots,h_{n+r})\in\mathbb{N}^{n+r}$ such that $h_1-j_1,\ldots,h_n-j_n,h_{n+1},\ldots,h_{n+r}$ are equal to 0 or 1 (independently) and such that $|H|=|J|+r$.

When J is a partition, the sequences H in (2.8) which are not partitions give functions S_H which are zero, and thus the summation restricts us to partitions H. For example, $S_{11}\cdot S_{22}=S_{3300}+S_{3210}+S_{2310}+S_{2211}$ and the term S_{2310}, which is zero, does not need to be written.

Finally, given two finite sets of variable $\mathbb{X}$ and $\mathbb{Y}$, one has the following property of factorisation which was an essential ingredient of classical elimination theory (but in terms of isobaric determinants; see, for a modern version, Berele and Regev (1987)). Let $J\in\mathbb{N}^n$, $\mathbb{X}$ be of cardinal n, $\mathbb{Y}$ of cardinal m. Then

$$S_{m^n+J}(\mathbb{X}-\mathbb{Y})=S_J(\mathbb{X})S_{m^n}(\mathbb{X}-\mathbb{Y})=S_J(\mathbb{X})\prod_{h,k}(x_h-y_k) \tag{2.9}$$

with $m^n+J=(j_1+m,\ldots,j_n+m)\in\mathbb{N}^n$.

3. Some Littlewood-type formulae

Consider the 32 formal expressions $\Pi_i(1\pm x_i)^{\pm1}\Pi_{i<j,i\leqslant j}(1\pm x_ix_j)^{\pm1}$ where the different symbols $\pm$ and $<$ or $\leqslant$ can be freely chosen (some of these expressions may represent the same function). Thanks to (2.3) with $\lambda=-1$ and (2.4), these formal expressions can be divided into eight sets, each of cardinal 4.

In Yang and Wybourne (1986, §§ 2, 4) one can find the S-function expansion of representatives of all the sets, except for the two sets containing respectively $\Pi_i(1-x_i)^{-1}\Pi_{i\leqslant j}(1-x_ix_j)^{-1}$ and $\Pi_i(1-x_i)\Pi_{i\leqslant j}(1-x_ix_j)$. The first one leads to large multiplicities and will not be discussed here. On the other hand, the second is multiplicity free and can be obtained as follows.

Littlewood (1950, ch XI) gives the expansion of $\Pi_{i\leq j}(1+z^2x_ix_j)$, and thus for $z=\sqrt{-1}$, the expansion of $\Pi_{i\leq j}(1-x_ix_j)$; in that case, it is convenient to index Schur functions by words rather than partitions. To be precise, given a word w in the alphabet $\{0, \alpha\}$, one takes its image by the application: $\{\text{words of degree } n\}\rightarrow\mathbb{N}^n$, $w=w_1,\ldots,w_n\rightarrow \mathbf{w}=(2w_1, 4w_2, \ldots, 2nw_n)$ with $\alpha=1$.

Then Littlewood's formula is

$$\prod_{i\leq j}(1-x_ix_j)=\sum(-1)^{|w|_\alpha}S_{\mathbf{w}}(\mathbb{X}) \tag{3.1}$$

sum on all words of degree n in $\{0,\alpha\}$, $|w|_\alpha$ denoting the degree of w in α. For example, for $n=3$, one has the words 000, $\alpha00$, $0\alpha0$, 00α, $\alpha\alpha0$, $\alpha0\alpha$, $0\alpha\alpha$, $\alpha\alpha\alpha$ and thus

$$\prod_{i\leq j\leq 3}(1-x_ix_j)=S_{000}-S_{200}-S_{040}-S_{006}+S_{240}+S_{206}+S_{046}-S_{246}$$

i.e. with partition indexing, $1-S_2+S_{31}-S_{411}-S_{33}+S_{431}-S_{442}+S_{444}$. Now, Pieri's formula (2.8) allows us to multiply by the factor $\Pi(1-x_i)$. We obtain words in $\{0, 1, \alpha, \beta=\alpha+1\}$ and Schur functions indexed by the image of these words through $w_i=\alpha\rightarrow 2i$; $w_i=\beta\rightarrow 2i+1$; $w_i=0\rightarrow 0$; $w_i=1\rightarrow 1$. Words having a factor $\ldots 01 \ldots$ or a factor $\ldots\alpha\beta\ldots$ need not be taken, as they correspond to zero Schur functions (being determinants having two identical columns). Let us call *good* a word with no factor $\ldots 01\ldots, \ldots\alpha\beta\ldots, \ldots 0\alpha\ldots, \ldots\beta 1\ldots$. Since $S_{\ldots 0\alpha\ldots}=-S_{\ldots\beta 1\ldots}$, we can eliminate any pair of words $w'0\alpha w''$ and $w'\beta 1 w''$, with w'' arbitrary and w' good. Finally, we are left with only good words, i.e. more concretely, with words which factor: $w=w'w''$, $w'\in\{1,\alpha\}^*$, $w''\in\{0,\beta\}^*$ (i.e. with a left factor in 1, α and a right factor in 0, β). To conclude, we have

$$\prod_{i\leq n}(1-x_i)\prod_{i\leq j\leq n}(1-x_ix_j)=\sum(-1)^{|w''|_\alpha}S_{\mathbf{w}'\mathbf{w}''}(\mathbb{X}) \tag{3.2}$$

sum on all words $w'w''$ of total degree n, $w'\in\{1,\alpha\}^*$, $w''\in\{0,\beta\}^*$. For example, for $n=3$, one has $(3+1)2^3$ such words; the words of degree 1 in α are $\alpha00$, $\alpha\beta0$, $\alpha0\beta$, $\alpha\beta\beta$, $\alpha10$, $1\alpha0$, $\alpha1\beta$, $1\alpha\beta$, 11α and furnish respectively the Schur functions S_{200}, S_{250}, S_{207}, S_{252}, S_{210}, S_{140}, S_{217}, S_{147}, S_{116}.

In terms of partitions, one can rewrite (3.2) as follows. Recall (Littlewood 1950, ch XI) that $\Pi_{i\leq j}(1+x_ix_j)=\Sigma_{i\in\Gamma}S_I(\mathbb{X})$, where Γ is the set of partitions of the type $(\gamma_1+1, \gamma_2+1, \ldots; \gamma_1, \gamma_2, \ldots)$ in Frobenius notation (Macdonald 1979, I.1). Then

$$\prod_{i\leq n}(1-x_i)\prod_{i\leq j\leq n}(1-x_ix_j)=\sum_I(-1)^{\varepsilon(I)}S_I(\mathbb{X}) \tag{3.3}$$

where the sum is over those partitions I for which one can remove one and only one vertical strip (Macdonald 1979, I.1) such that the resulting partition, denoted I^Γ, belongs to Γ. In that case, one puts $\varepsilon(I)=\frac{1}{2}|I^\Gamma|+|I|$.

4. The Cauchy formula and its applications

Let $\mathbb{X}=\{x_1,\ldots,x_n\}$, $\mathbb{Y}=\{y_1,\ldots,y_m\}$ be two sets of variables. Then one has the Cauchy formula:

$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_I S_I(\mathbb{X})S_I(\mathbb{Y}) \tag{4.1}$$

where the sum is over all partitions I. This formula can be deduced from the Binet–Cauchy formula for minors of a product of matrices.

Let $\varphi(x)=\Pi_{y\in\mathbb{Y}}(1-yx)$ be a polynomial such that $\varphi(0)=1$, the set $\mathbb{Y}$ being the set of roots of $\varphi(x^{-1})$, treated as a set of independent variables. From (4.1), one gets

$$\prod_{i=1}^{n}\varphi(x_i)^{-1}=\sum_I S_I(\mathbb{Y})S_I(\mathbb{X}) \tag{4.2}$$

and with the help of (2.3)-(2.6), the equivalent forms

$$\prod_{i=1}^{n}\varphi(x_i)=\sum_I(-1)^{|I|}S_I(\mathbb{Y})S_{I\sim}(\mathbb{X}) \tag{4.3}$$

$$\prod_{i=1}^{n}\varphi(\lambda x_i)=\sum_I(-\lambda)^{|I|}S_I(\mathbb{Y})S_{I\sim}(\mathbb{X}) \tag{4.4}$$

$$\prod_{i=1}^{n}\varphi(\lambda x_i)^{-1}=\sum_I\lambda^{|I|}S_I(\mathbb{Y})S_I(\mathbb{X}) \tag{4.5}$$

with λ any real scalar.

In all these formulae, the sum is over all partitions, but of course, since $S_I(\mathbb{Y})$ is zero if I has more than $\text{card}(\mathbb{Y})=\deg(\varphi)=m$ parts, the sum is limited to partitions with m parts at most. Thus, to compute explicitly (4.2), we can stick to the case where $\text{card}(\mathbb{X})=m$.

Let us treat the case $\varphi(x)=1-x^p$ with p a positive integer. Put $\mathbb{Y}=\{\zeta^0,\zeta,\ldots,\zeta^{p-1}\}$ with ζ a primitive p root of unity. Then the identity $(1-x^p)^{-1}=\Sigma\, x^{kp}$ implies that $S_n(\mathbb{Y})=1$ or 0 according to whether n is a multiple of p or not. More generally, for any $J=pH$, $H\in\mathbb{N}^p$ (i.e. such that each part of J is a multiple of p) $S_J(\mathbb{Y})=1$ since S_J is in that case the determinant of the identity matrix. The partitions I such that there exists $J=pH: S_I=\pm S_J$ are the partitions without a p core (Robinson 1961). Thus, finally

$$\prod_i(1-x_i^p)^{-1}=\sum S_J(\mathbb{X})=\sum\pm S_I(\mathbb{X}) \tag{4.6}$$

the first sum being on all $J=pH$, $H\in\mathbb{N}^p$ and the second on all partitions without a p core and with p parts at most.

For example, for $p=3$, one recovers the result of Yang and Wybourne (1986 § 5):

$$\prod_i(1-x_i^3)^{-1}=\sum S_J=\sum_{i,h\leqslant k}(S_{i+3k\; i+3h\; i}(\mathbb{X})-S_{i+3k+2\; i+3h+1\; i}(\mathbb{X})) \tag{4.7}$$

sum on all $J=(j_1,j_2,j_3)\in\mathbb{N}^3$ such that j_1,j_2,j_3 are multiples of 3.

From (4.3)-(4.5), one also gets the S-function series for $\Pi_i(1-x_i^p)$, $\Pi_i(1+x_i^p)$, $\Pi_i(1-x_i^p)^{-1}$.

Let now $\varphi(x)=1+x+\ldots+x^{p-1}$ and $\mathbb{Y}=\{\zeta,\ldots,\zeta^{p-1}\}$ with ζ a pth primitive root of unity. Then $1/\varphi(x)=(1-x)/(1-x^p)=\Sigma_k\, x^{kp}-x^{kp+1}$ shows that $S_j(\mathbb{Y})=1$, -1 or 0 according to $j\equiv 0$, 1 or $2,\ldots,p-1 \bmod p$; more generally, $S_I(\mathbb{Y})=\pm1$ or 0 according to the values of $i_1,\ldots,i_{p-1} \bmod p$. More precisely, up to a permutation of columns, the only non-zero determinants $S_H(\mathbb{Y})$, $H\in\mathbb{N}^{p-1}$, are the ones which factor in a determinant with 1 in the diagonal, -1 in the diagonal above, and a determinant with -1 in the diagonal, 1 in the diagonal below. Accordingly,

$$\prod_{i\leqslant n}(1+x_1+\ldots+x_i^{p-1})^{-1}=\sum_{0\leqslant r\leqslant p-1}(-1)^{p-1-r}\sum_H S_H(\mathbb{Y}) \tag{4.8}$$

sum on all $H\in\mathbb{N}^{p-1}$, $h_1,\ldots,h_{p-r-1}\equiv 0$, $h_{p-r},\ldots,h_{p-1}\equiv 1 \bmod p$.

5. Symmetrising operators

In the preceding sections, we have handled S series with only small multiplicities. The most efficient method, in our view, to generate more general series is to use symmetrising operators on the ring of functions of several variables (Lascoux and Schützenberger 1983). The oldest of these operators are the divided differences of Newton, mainly used in interpolation theory or numerical analysis, but also in the cohomology theory of flag manifolds (Bernstein *et al* 1973). The symmetrising operators allow us to increase the number of variables in the S series. We shall only use the total symmetriser π on the ring of series in $x_1, \ldots, x_n$, defined by

$$f \to \sum_{\mu \in \mathfrak{s}(n)} \left(f \cdot \prod_{i<j} (1 - x_j/x_i)^{-1} \right)^{\mu} = \pi(f) \tag{5.1}$$

denoting by f^{μ} the function $f(x_{\mu_1}, \ldots, x_{\mu_n})$.

Two basic properties of π are

$$f \text{ symmetrical} \Rightarrow \pi(fg) = f\pi(g) \tag{5.2}$$

$$\pi(x_1^{i_1} \ldots x_n^{i_n}) = S_I(\mathbb{X}) \qquad \text{when } I \in \mathbb{N}^n. \tag{5.3}$$

In fact, (5.3) is the definition by Jacobi of 'Schur functions' and has been extended by Weyl (the Weyl character formula) to groups other than $\mathrm{Gl}(n)$.

Schur functions are eigenfunctions of π, i.e. for $m \leqslant n$, $I \in \mathbb{N}^m$, $J \in \mathbb{N}^{n-m}$, one has

$$\pi[S_I(x_1 + \ldots + x_m) \cdot S_J(x_{m+1} + \ldots + x_n)] = S_{IJ}(\mathbb{X}) \tag{5.4}$$

where IJ is the element $i_1, \ldots, i_m, j_1, \ldots, j_{n-m}$ of $\mathbb{N}^n$.

From (5.4) with $m = n-1$, $I = j0\ldots0$, $J = 0$, one sees that

$$\pi\left(\prod_{i \leqslant i \leqslant n-1} (1 - x_i)^{-1} \right) = \prod_{1 \leqslant i \leqslant n} (1 - x_i)^{-1}. \tag{5.5}$$

More directly, writing

$$\prod_{1 \leqslant i \leqslant n-1} (1 - x_i)^{-1} = (1 - x_n) \prod_{1 \leqslant i \leqslant n} (1 - x_i)^{-1}$$

one can use (5.2) and recover (5.5):

$$\pi\left(\prod_{1 \leqslant i \leqslant n-1} (1 - x_i)^{-1} \right) = \pi(1 - x_n) \cdot \prod_{1 \leqslant i \leqslant n} (1 - x_i)^{-1} = \prod_{1 \leqslant i \leqslant n} (1 - x_i)^{-1}.$$

One checks without much more difficulty the following identities:

$$\pi\left(\prod_{1 \leqslant i \leqslant n-1} (1 - x_i x_n) \right) = \begin{cases} 1 & \text{if } n \text{ odd} \\ 1 - x_1 \ldots x_n & \text{if } n \text{ even} \end{cases} \tag{5.6}$$

$$\pi\left(x_n \prod_{1 \leqslant i \leqslant n-1} (1 - x_i x_n) \right) = \begin{cases} 0 & \text{if } n \text{ even} \\ x_1 \ldots x_n & \text{if } n \text{ odd} \end{cases} \tag{5.7}$$

$$\pi\left(x_n^2 \prod_{1 \leqslant i \leqslant n-1} (1 - x_i x_n) \right) = \begin{cases} (x_1 \ldots x_n)^2 & \text{if } n \text{ odd} \\ (x_1 \ldots x_n)^2 - x_1 \ldots x_n & \text{if } n \text{ even.} \end{cases} \tag{5.8}$$

We illustrate on a classical formula how to use the operator π. Let us consider

$$F(n-1) = \prod_{i \leqslant n-1} (1 - x_i)^{-1} \prod_{i<j \leqslant n-1} (1 - x_i x_j)^{-1}.$$

Then

$$\pi(F(n-1))=\pi((1-x_n)(1-x_1x_n)\ldots(1-x_{n-1}x_n)F(n))=(1-x_1\ldots x_n)F(n)$$

thanks to (5.2), (5.6) and (5.7). Assuming now by induction that $F(n-1)=\Sigma\, S_I(x_1+\cdots+x_{n-1})$, we transform this identity with the help of π into $(1-x_1\ldots x_n)F(n)=\Sigma\, S_I(x_1+\ldots+x_n)$, sum on all partitions $I=(i_1,\ldots,i_{n-1})$; dividing by $(1-x_1\ldots x_n)$, we finally obtain the identity of Schur (Macdonald 1979, I.5):

$$\prod_{i\leqslant n}(1-x_i)^{-1}\prod_{i<j\leqslant n}(1-x_ix_j)^{-1}=\sum S_J(x_1+\ldots+x_n). \tag{5.9}$$

The other formulae of this type given by Littlewood could be similarly proven; here is a new candidate:

$$\prod_{i\leqslant n}(1-x_i+x_i^2)^{-1}\prod_{i<j\leqslant n}(1-x_ix_j)^{-1}=\sum \pm m_I S_{I^\sim}(\mathbb{X}) \tag{5.10}$$

sum on all partitions $I=\ldots 3^{m_3}2^{m_2}1^{m_1}$ such that $m_{2j+1}\equiv 0$ or $1 \bmod 3$, with $m_I=(m_2+1)(m_4+1)(m_6+1)\ldots$, the sign $\pm$ being $(-1)^{[m_1/3]+[m_3/3]+\cdots}$, with $[m/3]$ the integral part of $m/3$.

Proof. Let $G_n(k)$ denote the function $\Sigma \pm m_I S_{I^\sim}(x_1+\ldots+x_k)$, with the m_I defined above. From (5.6)-(5.8), we have

$$\pi\Big((1-x_n+x_n^2)\prod_{i\leqslant n-1}(1-x_ix_n)\Big)=\begin{cases}1-x_1\ldots x_n+(x_1\ldots x_n)^2 & \text{if } n \text{ odd}\\ (1-x_1\ldots x_n)^2 & \text{if } n \text{ even.}\end{cases}$$

Suppose that $G_{n-1}(n-1)=\Sigma \pm m_I S_{I^\sim}(x_1+\ldots+x_{n-1})$. The image of this equality by π is

$$G_{n-1}(n)=\begin{cases}G_n(n)\dfrac{1+(x_1\ldots x_n)^3}{1+x_1\ldots x_n} & \text{if } n \text{ odd}\\ G_n(n)(1-x_1\ldots x_n)^2 & \text{if } n \text{ even.}\end{cases}$$

This determines inductively $G_n(n)$; we just illustrate for the step $n=3\to n=4$ that $G_n(n)$ is as stated in (5.10). Starting from $G_3(3)=\Sigma \pm(h-j+1)S_{khj}(x_1+x_2+x_3)$, $j\equiv 0$ or 1, $k-h\equiv 0$ or $1 \bmod 3$, we get with the help of π: $(1-x_1x_2x_3x_4)^2G_4(4)=G_3(4)=\Sigma \pm(h-j+1)S_{khj}(x_1+x_2+x_3+x_4)$ and thus

$$\begin{aligned}G_4(4)&=\sum \pm(i+1)(h-j+1)(x_1x_2x_3x_4)^iS_{khj}(x_1+x_2+x_3)\\&=\sum \pm(i+1)(h-j+1)S_{i+k\,i+h\,i+j\,i}(x_1+x_2+x_3+x_4).\end{aligned}$$

Instead of computing the Schur functions for the set of roots of a given polynomial $\varphi(x)$, as in § 4, one can also use the symmetriser π. Take for example $\varphi(x)=1-x+x^2-x^3$. Then, by direct development,

$$\begin{aligned}1/\varphi(x_1)\varphi(x_2)&=(1+x_1)(1+x_2)/(1-x_1^4)(1-x_2^4)\\&=\sum_{h,k}S_{kh}(x_1+x_2)-S_{k\,h+1}(x_1+x_2)-S_{k+1\,h}(x_1+x_2)+S_{k+1\,h+1}(x_1+x_2)\end{aligned}$$

sum on all pair of positive integers, $h, k\equiv 0 \bmod 4$. The image of $1/\varphi(x_1)\varphi(x_2)$ by the operator π corresponding to $\mathbb{X}=\{x_1,x_2,x_3\}$ is

$$\pi(1-x_3+x_3^2-x_3^3)/\varphi(x_1)\varphi(x_2)\varphi(x_3)=(1-x_1x_2x_3)/\varphi(x_1)\varphi(x_2)\varphi(x_3).$$

Thus

$$1/\varphi(x_1)\varphi(x_2)\varphi(x_3) = \sum_{i,h,k} S_{i+k\,i+h\,i}(\mathbb{X}) - S_{i+k\,i+h+1\,i}(\mathbb{X}) - S_{i+k+1\,i+h\,i}(\mathbb{X}) + S_{i+k+1\,i+h+1\,i}(\mathbb{X}) \tag{5.11}$$

sum on all triple $i, h, k \in \mathbb{N}^3$, $h, k \equiv 0 \bmod 4$.

Moreover, since for $n \geqslant 4$ one has $\pi(1-x_n+x_n^2-x_n^3)=1$, π being the operator corresponding to $\mathbb{X}=\{x_1,\ldots,x_n\}$, formula (5.11) gives the expansion of $1/\varphi(x_1)\ldots\varphi(x_n)$ for general n.

6. Other methods to generate S-function series

In the preceding section, we did not use the fact that S functions are compatible with addition (formula (2.7)) and composition (as operators on the ring of polynomials); one can also combine S functions with other symmetrical functions also considered as operators.

Doing so, one can thus give a more general outlook to all the formulae written in the preceding sections and produce apparently new ones.

For example, (5.9) should be written, with $\mathbb{P}$ any polynomial,

$$\sum_J z^{|J|} S_J(\mathbb{P}) = \sum_j z^j S_j(\mathbb{P} + S_{11}(\mathbb{P})). \tag{6.1}$$

Taking $\mathbb{P}=(x_1+\ldots+x_n)-(y_1+\ldots+y_m)=\mathbb{X}-\mathbb{Y}$ gives (using $S_{11}(\mathbb{X}-\mathbb{Y})=S_{11}(\mathbb{X})-\mathbb{X}\mathbb{Y}+S_2(\mathbb{Y})$)

$$\sum_J z^{|J|} S_J(\mathbb{X}-\mathbb{Y}) = \sum_j z^j S_j(\mathbb{X}-\mathbb{Y}+S_{11}(\mathbb{X})-\mathbb{X}\mathbb{Y}+S_2(\mathbb{Y}))$$

$$= \frac{\prod_{j\leqslant m}(1-zy_j)\prod_{i\leqslant n, j\leqslant m}(1-zx_iy_j)}{\prod_{i\leqslant n}(1-zx_i)\prod_{i<j\leqslant n}(1-x_ix_j)\prod_{i\leqslant j\leqslant m}(1-y_iy_j)}. \tag{6.2}$$

Similarly, the Cauchy formula (4.1) becomes, through $\mathbb{X}\to\mathbb{X}-\mathbb{X}'$, $\mathbb{Y}\to\mathbb{Y}-\mathbb{Y}'$,

$$\sum z^{|J|} S_J(\mathbb{X}-\mathbb{X}')S_J(\mathbb{Y}-\mathbb{Y}') = \sum z^j S_j(\mathbb{X}\mathbb{Y}-\mathbb{X}'\mathbb{Y}-\mathbb{X}\mathbb{Y}'+\mathbb{X}'\mathbb{Y}')$$

$$= \frac{\prod_{x'\in\mathbb{X}', y\in\mathbb{Y}}(1-x'y)\prod_{x\in\mathbb{X}, y'\in\mathbb{Y}'}(1-xy')}{\prod_{x\in\mathbb{X}, y\in\mathbb{Y}}(1-xy)\prod_{x'\in\mathbb{X}', y'\in\mathbb{Y}'}(1-x'y')}.$$

Decomposing according to $\mathbb{X}'$ and $\mathbb{Y}'$ and comparing the coefficients of $S_H(\mathbb{X}')\cdot S_K(\mathbb{Y}')$ in both members, one gets, for every pair of partitions H, K (Lascoux and Schützenberger 1985b, Macdonald 1985),

$$\sum_J S_{J/K}(\mathbb{X})S_{J/H}(\mathbb{Y}) = \sum_j S_j(\mathbb{X}\mathbb{Y}) \sum_I S_{K/I}(\mathbb{X})S_{H/I}(\mathbb{Y}). \tag{6.3}$$

As for the composition of S functions as operators, called a plethysm by Littlewood (who denotes $S_I(S_J(\mathbb{X}))$ by $J\otimes I$), it is still one of the most important problems in classical group theory to find efficient algorithms or combinatorial objects to describe it; § 3 is a variation about $S_j(S_2)$, $S_j(S_{11})$, $S_{1^j}(S_2)$, $S_{1^j}(S_{11})$). One can use the symmetrising operators, as in § 5, to get recursions concerning the plethysm. However, it must be noted that the theory of recursive sequences (or of Padé approximants), allowing us to express, from the expansion of a rational function $P(z)/Q(z)$, both $P(z)$ and

$Q(z)$, provides *closed* expressions for the plethysm, though not very useful for computations. For example, by definition,

$$\sum_r z^r S_r(S_{111}(\mathbb{X})) = \prod_{1\leq i<j<h\leq n} (1-zx_ix_jx_h)^{-1}.$$

Let us write T_r for $S_{rrr}(\mathbb{X})$, let $m=\binom{n}{3}$, $I=(i_1,\ldots,i_m)$ be a partition and T_I be the determinant $|T_{i_{m+1-k}+k-h}|_{1\leq h,k\leq m}$. Then, for any integer $q\geq m-1$, one has

$$S_I(S_{111}(\mathbb{X})) = T_{q^m+I}/T_{q^m} \tag{6.4}$$

q^m+I meaning the partition $q+i_1,\ldots,q+i_m$.

Proof. Let $n=\mathrm{card}(\mathbb{X})\geq 3$. We then have

$$(1-zx_1x_2x_3)^{-1} = \sum z^r S_{rrr}(x_1+x_2+x_3). \tag{6.5}$$

Writing $(1-x_1x_2x_3)^{-1} = Q/\Pi_{1\leq i<j<h\leq n}(1-zx_ix_jx_h)^{-1}$, we see that the image of (6.5) by the operator π is

$$\sum z^r S_{rrr}(\mathbb{X}) = \sum z^r T_r = \pi(Q)\left(\prod_{1\leq i<j<h\leq n}(1-zx_ix_jx_h)^{-1}\right)^{-1}.$$

Now, the factorisation formula (2.9) allows us, given the expansion of a rational series, to express symmetric functions of the poles (or of the roots): if $\pi(Q)=\Pi_{y\in\mathbb{Y}}(1-zy)$, then

$$S_{q^m+I}(S_{111}(\mathbb{X})-\mathbb{Y}) = T_{q^m+I} = S_{q^m}(S_{111}(\mathbb{X})-\mathbb{Y})\cdot S_I(S_{111}(\mathbb{X})).$$

The same result as (6.4) holds for $S_I(S_{1^p}(\mathbb{X}))$, the function T_r this time being $S_{r^p}(\mathbb{X})$ and m being $\binom{n}{p}$.

Finally, we refer to Macdonald (1979) for the other bases of symmetric functions. We shall just mention a formula of Gordan (1899), also found in Lascoux and Schützenberger (1985b, proposition 1.9), concerning the multiplication of a Schur function by a monomial function (formula (2.8) is, in fact, the special case of the monomial function S_{1^r}). We detail here only the multiplication of $S_{0\ldots 0}=1$ by a monomial function: for any $I\in\mathbb{N}^n$, then

$$\sum_{\mu\in\mathfrak{S}(n)} (x_1^{i_1}\ldots x_n^{i_n})^\mu = \sum_{\mu\in\mathfrak{S}(n)} S_{I^\mu}(\mathbb{X}). \tag{6.6}$$

For example, writing in short $x^{i_1\cdots i_n}$ and S_H, one has, for $n=4$ and $I=0099$, $x^{0099}+x^{0909}+x^{9009}+x^{0990}+x^{9090}+x^{9900}=S_{0099}+S_{0909}+S_{9009}+S_{0990}+S_{9900}$. Expanding any product $\Pi_i\,\varphi(x_i)$ in terms of monomial functions, we easily deduce from (6.6) the expansion of this product in terms of the Schur function; for example $(1+x_1^9)\ldots(1+x_4^9)=\Sigma_H S_H(x_1+\ldots+x_4)$, sum on the 2^4 4-tuples of integers $H=(h_1,\ldots,h_4)$, $h_1,\ldots,h_4=0$ or 9. In this way, one obtains formulae (27)-(35*b*) of Yang and Wybourne (1988) who limit themselves to power 4.

References

Berele A and Regev A 1987 *Adv. Math.* **64** 118

Bernstein I N, Gelfand I M and Gelfand S 1973 *Usp. Math. Nauk* **28** 3

Cummins C J and King R C 1987 *Some noteworthy S-function identities. Preprint*

Einstein A and Zweistein B 1903 *Z. Natür. Phys.* **18** 199

Gordan P 1899 *Math. Ann.* **52** 501
King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
Lascoux A and Schützenberger M P 1983 *Springer Lecture Notes in Mathematics* **996** 118
—— 1985a *Lett. Math. Phys.* **10** 111
—— 1985b *Formulaire Raisonné de Fonctions Symétriques* (Paris: Publications Université Paris 7)
Littlewood D E 1950 *The Theory of Group Characters* (Oxford: Oxford University Press) 2nd edn
Macdonald I G 1979 *Symmetric Functions and Hall polynomials* (Oxford: Clarendon) (Russian transl. by Zelevinsky A 1985 (Moscow: Mir) with extra formulae)
Muir T 1906-23 *History of Determinants* (London: Blackie)
Robinson G de B 1961 *Representation Theory of the Symmetric Group* (Toronto: University of Toronto Press)
Sylvester J J 1973 *Collected Work* (New York: Chelsea)
Yang M and Wybourne B G 1986 *J. Phys. A: Math. Gen.* **19** 3513